J. Appl. Maths Mechs Vol. 56, No. 4, pp. 518–526, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

THE POSITIONS OF RELATIVE EQUILIBRIUM IN THE CIRCULAR ORBIT OF AN ELASTIC ARTIFICIAL SATELLITE AND THEIR STABILITY†

S. V. CHAIKIN

Irkutsk

(*Received* 15 *April* 1991)

The problem of the stability of motion in a circular orbit in a central Newtonian field of force of an elastic artificial satellite, modelled as a free solid body with an arbitrary elastic section connected to it, is considered in a limited formulation. Using Routh's theorem and assuming that the vector of small deformations of the elastic section can be represented in the form of a finite series in the known eigenmodes of its free oscillations [1] and certain others, the positions of relative equilibrium of the spacecraft (numbering 24) are obtained. The positions of equilibrium in which the elastic section is deformed are found approximately. The sufficient conditions for the stability of the positions of equilibrium obtained are found, and the necessary and sufficient conditions that the elastic section should be undeformed in a position of equilibrium are indicated. An example of an artificial satellite modelled as a solid body with an arbitrary rectilinear elastic rod attached to it is considered.

1. FORMULATION OF THE PROBLEM

CONSIDER the motion in a central Newtonian field of force of a spacecraft modelled as a solid body with an elastic section attached to it. We will assume that we can neglect the effect of the motion of the spacecraft relative to its centre of mass on the displacement of the latter in a Kepler circular orbit of radius R with constant angular velocity ω around an attracting centre.

To describe the motion we will introduce the following right rectangular Cartesian systems of coordinates: $Oy_1y_2y_3$ is the orbital system of coordinates with a pole at the centre of mass of the satellite, denoted henceforth by the point O, the Oy_3 axis with the vector γ is directed along the radius-vector of the point O with respect to the attracting centre, Oy_2 and Oy_1 with unit vectors β and α, respectively, are directed along the binormal to the plane of the orbit and along its transversal at the point O in the direction of motion of the centre of mass of the satellite, $O_1x_1x_2x_3$ is a system of coordinates rigidly connected to the satellite, the pole O_1 of which is placed at the centre of mass, while the axes are directed along the principal central axes of the undeformed satellite, $\mathbf{i}^k$ is the unit vector along the O_1x_k axis, $Ox_1x_2x_3$ is a system of coordinates with a pole at the centre of mass of the satellite and unit vectors of the axes $\mathbf{i}^k$, respectively, Ω is the angular velocity of the trihedron $Ox_1x_2x_3$ with respect to $Oy_1y_2y_3$, and $\omega = \omega\beta$ is the vector of the orbital angular velocity of the orbital system of coordinates.

We will define the positions of relative equilibrium of the spacecraft as the state of rest with respect to the orbital system of coordinates. If the elastic section is in a deformed state in a position of relative equilibrium, the position of equilibrium will be called non-trivial.

Suppose the points of the solid of the satellite occupy the region $V_1 \subset R^3$, the points of the elastic section in the undeformed state occupy the region $V_2 \subset R^3$, Γ is the common boundary of the regions V_1 and V_2 and $V = V_1 \cup V_2$. We will assume that the regions are specified in $O_1x_1x_2x_3$;

† *Prikl. Mat. Mekh.* Vol. 56, No. 4, pp. 615–623, 1992.

$\mathbf{r} = x_1\mathbf{i}^1 + x_2\mathbf{i}^2 + x_3\mathbf{i}^3$ is the radius-vector of an arbitrary point of the spacecraft in the natural state with respect to the point O_1 and $\mathbf{u}(t, \mathbf{r})$ is the vector of small deformations of the point defined by the vector $\mathbf{r}$. The function $\mathbf{u}$: $(t, \mathbf{r}) \to \mathbf{u}(t, \mathbf{r}) \in R^3$ possesses sufficient smoothness with respect to t and $\mathbf{r}$, $t \in [t_0, \infty)$, $\mathbf{r} \in V$; $\mathbf{u}(t, \mathbf{r}) = 0$ when $\mathbf{r} \in V_1$. The radius vector of the centre of mass of the elastic satellite with respect to point O_1 (m is the mass of the spacecraft)

$$\rho \equiv m^{-1} \int_V (\mathbf{r} + \mathbf{u})\, dm = m^{-1} \int_V \mathbf{u} dm$$

We will indicate the assumptions under which we consider the motion of the satellite.

Assumption 1. The vector of small deformation of the points of the elastic section of the spacecraft in certain orthogonal local systems of coordinates with unit vectors $\mathbf{f}_k$ ($k = 1, 2, 3$) can be represented by a finite series [1]

$$\mathbf{u}(t, \mathbf{r}) = \sum_{n=1}^{h} (q_n^1 w_n \mathbf{f}_1 + q_n^2 u_n \mathbf{f}_2 + q_n^3 v_n \mathbf{f}_3) = \sum_{n=1}^{N} q_n(t)\, \psi_n(\mathbf{r}) \tag{1.1}$$

where $w_n(\mathbf{r})$, $u_n(\mathbf{r})$, $v_n(\mathbf{r})$ are natural modes of the free elastic oscillations of the section in a local system of coordinates and $q_n^k(t)$ are generalized coordinates corresponding to the natural forms $\psi_n (N = 3n)$.

Assumption 2. Taking representation (1.1) into account in the expression for the inertia tensor of the satellite about the point O

$$\mathbf{J} \equiv \int_V ((\mathbf{r} + \mathbf{u} - \rho)E - (\mathbf{r} + \mathbf{u} - \rho) : (\mathbf{r} + \mathbf{u} - \rho))\, dm$$

where $a:b$ is the diadic product of the vector a and b, we will neglect terms that are non-linear in q_n, i.e. we will assume that

$$J \equiv J_0 + \sum_{n=1}^{N} q_n J_n \tag{1.2}$$

where $\mathbf{J}_0$ is the inertia tensor of the satellite in the undeformed state with respect to the point O_1, E is the unit tensor and

$$\mathbf{J}_n = \int_{V_2} (2\mathbf{r}\psi_n E - \psi_n : \mathbf{r} - \mathbf{r} : \psi_n)\, dm \tag{1.3}$$

Assumption 3. The central ellipsoid of inertia of the spacecraft in the undeformed state is triaxial.

Assumption 4. We will take as the potential energy of the force of gravitational attraction Π_g its approximate expression calculated to within terms of the order of $L_3 R^{-3}$, where L is the characteristic linear dimension of the spacecraft [1]

$$\Pi_g = -\frac{\mu m}{R} + \frac{1}{2}\omega^2 (3\gamma \mathbf{J} \gamma - \operatorname{tr} \mathbf{J}) \tag{1.4}$$

Assumption 5. We will represent the potential energy of an isotropic elastic section for small deformations [3, 4]

$$\Pi = \frac{1}{2} \int_{V_2} \sum_{i,j=1}^{3} \epsilon_{ij} \sigma_{ij} dv = \frac{1}{2} \int_{V_2} \sum_{m,n,k,l=1}^{3} a_{mnkl} \epsilon_{mn} \epsilon_{kl} dv$$

taking (1.1) into account in the form

$$\Pi = \frac{1}{2} \sum_{n,m=1}^{N} c_{nm} q_n q_m \tag{1.5}$$

where c_{nm}, a_{mnkl} are constant coefficients and the $N\times N$ matrix $C\equiv\|c_{nm}\|$ is positive definite, and ϵ_{ij} and σ_{ij} are components of the tensor of the infinitely small strain and stress in a local system of coordinates.

The different approaches when setting up the equation of motion of complex mechanical systems are described in [2, 3] and, in the case considered, with assumptions 1–4, are obtained, for example, from the equations in [3].

Note. Assumption 2 corresponds formally to the fact that in the equations of motion we are neglecting terms that are non-linear in q_n and q_m, and their derivatives.

We know [2–4] that the equations of motion of the spacecraft in the case considered, in addition to partial integrals of the direction cosines

$$U_1\equiv\gamma\gamma-1=0,\quad U_2\equiv\beta\beta-1=0,\quad U_3\equiv\gamma\beta=0 \tag{1.6}$$

admit of an integral of the Jacobi type

$$U\equiv\tfrac{1}{2}\mathbf{\Omega J\Omega}+\mathbf{\Omega G}+T+\Pi+\Pi_g-\tfrac{1}{2}\omega\mathbf{J}\omega=\text{const} \tag{1.7}$$

where $\mathbf{G}$ and T are the vector of the kinetic momentum about the point O and the kinetic energy of the satellite in its relative motion.

Using (1.1) we obtain

$$\mathbf{G}\equiv\int_V(\mathbf{r}+\mathbf{u}-\rho)\times(\mathbf{u}^{\cdot}-\rho^{\cdot})\,dm=\sum_{n=1}^{N}q_n^{\cdot}(\mathbf{G}_n+\sum_{m=1}^{N}\mathbf{G}_{mn}q_m) \tag{1.8}$$

$$T\equiv\frac{1}{2}\int_V(\mathbf{u}^{\cdot}-\rho^{\cdot})^2dm=\frac{1}{2}\sum_{n,m=1}^{N}a_{nm}q_n^{\cdot}q_m^{\cdot} \tag{1.9}$$

$$(\quad)^{\cdot}\equiv\partial(\quad)/\partial t,\qquad \mathbf{G}_n\equiv\int_V\mathbf{r}\times(\psi_n-m^{-1}\int_V\psi_n dm)\,dm$$

$$\mathbf{G}_{np}\equiv\int_V(\psi_n-m^{-1}\int_V\psi_n dm)\times(\psi_p-m^{-1}\int_V\psi_p dm)\,dm$$

$$a_{np}\equiv\int_V(\psi_n-m^{-1}\int_V\psi_n dm)(\psi_p-m^{-1}\int_V\psi_p dm)\,dm$$

From Routh's theorem [5], if values of the variables

$$\mathbf{\Omega}=\mathbf{\Omega}^0,\quad \beta=\beta^0,\quad \gamma=\gamma^0,\quad q_n=q_n^0,\quad q_n^{\cdot}=q_n^{0\cdot}\quad (n=1,2,\ldots,N) \tag{1.10}$$

exist which give the integral U an isolated minimum for fixed values of the integrals U_i, then these values, generally speaking, will correspond to one of the real motions of the satellite and this motion will be stable with respect to Ω, β, γ, $q_1,\ldots,q_N$, $q_1^{\cdot},\ldots,q_N^{\cdot}$.

2. POSITIONS OF RELATIVE EQUILIBRIUM

Suppose the new variables

$$\mathbf{\Omega}^*\equiv\mathbf{\Omega}-\mathbf{\Omega}^0,\quad \beta^*\equiv\beta-\beta^0,\quad \gamma^*\equiv\gamma-\gamma^0,\quad q_n^*\equiv q_n-q_n^0,\quad q_n^{*\cdot}\equiv q_n^{\cdot}-q_n^{0\cdot}$$

To determine the values of (1.10), which give the integral (1.7) stationary values under conditions (1.6), we will use the method of undetermined Lagrange multipliers

$$W\equiv U+3\omega^2\lambda(q^0)U_3-\tfrac{3}{2}\omega^2\sigma(q^0)U_1+\tfrac{1}{2}\omega^2\nu(q^0)U_2 \tag{2.1}$$

Here $q^*\equiv(q_1^*,\ldots,q_N^*)^T$, $q^{*\cdot}=(q_1^{*\cdot},\ldots,q_N^{*\cdot})$ etc., $\lambda(q^0)$, $\nu(q^0)$, $\sigma(q^0)$ are undetermined Lagrange multipliers and the sign T denotes transposition. The equations for finding the quantities (1.10) and the undetermined multipliers can be obtained from the equation

$$\delta W=0\quad\text{for}\quad \mathbf{\Omega}^*=0,\ \beta^*=0,\ \gamma^*=0,\ q^*=0,\ q^{*\cdot}=0.$$

They can be written as follows:

$$\gamma^0\gamma^0 - 1 = 0, \quad \beta^0\beta^0 - 1 = 0, \quad \gamma^0\beta^0 = 0 \tag{2.2}$$

$$3\omega^2((J - \sigma E)\gamma^0 + \lambda\beta^0) = 0$$
$$\omega^2((\nu E - J)\beta^0 + 3\lambda\gamma^0) = 0 \tag{2.3}$$

$$\frac{1}{5}\,\Omega^0 \mathbf{J}_n \Omega^0 + \Omega^0 \sum_{m=1}^{N} \mathbf{G}_{nm} q_m^{0\cdot} + \sum_{m=1}^{N} c_{nm} q_m^0 - 0{,}5\,\omega^2 \beta^0 \mathbf{J}_n \beta^0 +$$
$$+ \frac{3}{2}\,\omega^2 \gamma^0 \mathbf{J}_n \gamma^0 - \frac{1}{2}\,\omega^2 \operatorname{tr} \mathbf{J}_n = 0 \quad (n = 1, \ldots, N) \tag{2.4}$$

$$\mathbf{J}\Omega^0 + \sum_{m=1}^{N} \mathbf{G}_m^0 q_m^{0\cdot} = 0, \quad \mathbf{G}_n^0 \Omega^0 + \sum_{m=1}^{N} a_{nm} q_m^{0\cdot} = 0 \tag{2.5}$$

$$\mathbf{J} \equiv \mathbf{J}(q^{\boldsymbol{\cdot}} = 0) \equiv \mathbf{J}(q^0) = \mathbf{J}_0 + \sum_{n=1}^{N} q_n^0 \mathbf{J}_n, \quad \mathbf{G}_p^0 \equiv \mathbf{G}_p + \sum_{m=1}^{N} \mathbf{G}_{mp} q_m^0$$

The quantities $\mathbf{G}_n$ and $\mathbf{G}_{mp}$ are defined in (1.8).

Note. The function W is a combination of the integrals (1.6) and (1.7) written in perturbations with respect to the unperturbed values (1.10). The writing of the integrals (1.6) and (1.7) and Eqs (2.2)–(2.5) in tensor form indicates the possibility of choosing the system of coordinates that is most convenient for solving (2.2)–(2.5) and for investigating the conditions of stability of the steady-state motions obtained.

We will denote the system of coordinates with pole at the point O, in which the matrix of the components of the inertia tensor $\mathbf{J}(q^0)$ is diagonal, by $Ox_1^0x_2^0x_3^0$, and we will suppose that $\mathbf{e}^k(q^0)$ are the unit vectors of the corresponding axes and $p(q^0)$ is the orthogonal matrix of the transition form $(Ox_1x_2x_3)$ and $Ox_1^0x_2^0x_3^0$.

For the matrices of the component of the tensors encountered in the present paper, unless stated otherwise, in the axes $Ox_1^0x_2^0x_3^0$ we will use the same notation but without distinguishing them by boldface type.

The kinetic energy T_0 of the satellite in its motion about the centre of mass is a positive definite quadratic form with symmetrical matrix $D(q)$ with respect to the quasi-velocities $(\Omega + \omega)$ and the generalized velocities $q^{\boldsymbol{\cdot}}$, which vanish when $\Omega + \omega \equiv 0$, $q^{\boldsymbol{\cdot}} \equiv 0$

$$T_0 = \frac{1}{2}\,(\Omega^T + \omega, q^{T\cdot})\; D(q) \begin{pmatrix} \Omega + \omega \\ q^{\cdot} \end{pmatrix}$$

$$D(q^0) \equiv \| d_{ij} \| = \begin{Vmatrix} J(q^0) & G_1^0 & \ldots & G_N^0 \\ G_1^{0T} & a_{11} & \ldots & a_{1N} \\ \vdots & \vdots & & \\ G_N^{0T} & a_{N1} & \ldots & a_{NN} \end{Vmatrix}$$

and consequently $\Omega^0 \equiv 0$, $q^{0\boldsymbol{\cdot}} \equiv 0$ is the unique solution of (2.5) since $\det D(q^0) \neq 0$.

Multiplying the first equation of (2.3) by β^T and the second equation by γ^T, and taking (2.2) and the equation $J = J^T$ into account, we obtain $\lambda(q^0) = -\beta^T J\gamma$, $\lambda(q^0) = \beta^T J\gamma$, whence we conclude that $\lambda(q^0) \equiv 0$ while $(\sigma(q^0), \gamma)$ and $(\nu(q^0), \beta)$ are natural pairs of $J(q^0)$.

We will determine the eigenvalues $\mu^k(q^0)$ and the eigenvectors $\mathbf{e}^k(q^0)$ of the matrix $J(q^0)$ by the method of perturbation theory (see, for example, [6]), apart from terms that are linear in q_n. With the same accuracy as in Assumption 2 we will determine the inertia tensor of the spacecraft from formula (1.2).

Suppose I, I_0 and I_n are symmetrical matrices of the components $\mathbf{J}$, $\mathbf{J}_0$ and $\mathbf{J}_n$, respectively, in the $Ox_1x_2x_3$ system.

Then

$$\mu^k(q^0) = \mu^k + \sum_{n=1}^{N} q_n^0 \mu_n^k, \quad \mu_n^k = I_n^{kk}, \tag{2.6}$$

$$\mathbf{e}^k(q^0) = \mathbf{i}^k + \sum_{n=1}^{N} q_n^0 \mathbf{i}_n^k, \quad \mathbf{i}_n^k = \sum_{j=1(j\neq k)}^{3} (\mu^k - \mu^j)^{-1} I_n^{jk} \mathbf{i}^j \tag{2.7}$$

where $(\mu^k, \mathbf{i}^k)$ is the natural pair of the matrix $I_0 \equiv \mathrm{diag}(\mu^1, \mu^2, \mu^3)$.

In the $Ox_1x_2x_3$ system the column of components i^k of the vector $\mathbf{i}^k$ is obviously equal to $i^k = (i_1^k, i_2^k, i_3^k)$, $i_j^k = \delta_{jk}$, where δ_{jk} is the Kronecker delta, while the column of components $e^k(q_0)$ of the vector $e^k(q^0)$ in $Ox_1x_2x_3$ is defined by (2.7). The orthogonal transfer matrix (calculated apart from terms that are linear in q_n^0) $P(q^0) \equiv (e^1(q^0), e^2(q^0), e^3(q^0))$, while by the theorem on the conversion of the components of tensors on changing to a new system of coordinates

$$J(q^\bullet) \equiv P(q^0)(I_0 + \sum_{n=1}^{N} q_n^0 I_n + \sum_{n=1}^{N} q_n^\bullet I_n) P^T(q^0)$$

$$J(q^\bullet = 0) \equiv J(q^0) \equiv P(q^0)(I_0 + \sum_{n=1}^{N} q_n^0 I_n) P^T(q^0) = \mathrm{diag}(\mu^1(q^0), \mu^2(q^0), \mu^3(q^0))$$

$$J_n \equiv P(q^0) I_n P^T(q^0)$$

The solution of Eqs (2.1)–(2.5) in projections on the $Ox_1^0x_2^0x_3^0$ axes can be written as follows:

$$\forall k, l, m \in \{1, 2, 3\}, \quad k \neq l \neq m$$

$$\Omega_i^0 = 0, \quad q_n^{0\cdot} = 0, \quad q_n^0 = C^{-1} M$$

$$\lambda(q^0) \equiv 0, \quad \nu(q^0) = \mu^k(q^0), \quad \beta_i^0 = \pm\delta_{ik}, \sigma(q^0) = \mu^m(q^0), \ \gamma_i^0 = \pm\delta_{im} \tag{2.8}$$

where $M = \omega^2 (M_1, \ldots, M_N)^T$, $M_n = \mu_n^k - \mu_n^m + \tfrac{1}{2}\mu_n^l$, and due to the smallness of q_n^0 in (2.4) we have assumed $I_n = I_n$ $(n = 1, \ldots, N)$.

In order to obtain the values of the variables Ω^0, β^0, γ^0 in the $Ox_1x_2x_3$ system, it is sufficient to use the matrix $P^{-1}(q^0)$.

Hence, we have proved the following assertion.

Assertion 2.1. With Assumptions 1–5, in order that a trivial position of relative equilibrium ($q_n^0 = 0$) should exist it is necessary and sufficient that

$$\forall n = 1, \ldots, N, \quad \mu_n^m - \mu_n^k - 0{,}5\,\mu_n^l = 0.$$

Assertion 2.2. With Assumptions 1–5, the set of positions of relative equilibrium (2.8) of an elastic satellite, with respect to the principal central axes, constructed for a given position of equilibrium, is characterized by the fact that these axes are directed along the axes of the orbital system of coordinates and in this sense the set of positions of relative equilibrium of the elastic satellite coincides with that for a solid [7].

It follows from (2.8) that the number of different positions of equilibrium of the spacecraft in a circular orbit is 24. The set of positions of equilibrium of an elastic spacecraft can be divided into six groups with four positions of equilibrium each, if we specify for a group, for example, the direction of the vector $\mathbf{e}^3(q^0)$ along one of the axes of the orbital system of coordinates in the positive or negative directions. The four positions of equilibrium which thereby occur in each defined group are specified by the direction of the vector $\mathbf{e}^2(q^0)$ along any other axis of the orbital system of coordinates $(\mathbf{e}^3(q^0) \perp \mathbf{e}^2(q^0))$ in the positive or negative directions.

3. STABILITY OF THE POSITION OF EQUILIBRIUM

To obtain the conditions for the relative positions of equilibrium obtained to be stable we will use the standard method (see, for example, [5]). We will introduce the following notation:

$$y_1 = \gamma_1^*, \quad y_2 \equiv \beta_1^*, \quad y_3 \equiv \gamma_2^*, \quad y_4 \equiv \beta_2^*, \quad y_5' \equiv \gamma_3^*, \quad y_6 \equiv \beta_3^*$$

$$y_{6+1} \equiv q_1^*, \ldots, y_{6+N} \equiv q_N^*$$

$$y_{7+N} \equiv \Omega_1^*, \ldots, y_{9+N} \equiv \Omega_3^*, \quad y_{9+N+1} \equiv q_1^{*\cdot}, \ldots, y_{9+2N} \equiv q_N^{*\cdot}$$

$$A_* \equiv \left\| \frac{\partial W}{\partial y_i \partial y_j} \right\|_{i,j=1}^{i,j=9+2N}, \quad B_* \equiv \| b_{ij} \| = \left\| \frac{\partial U_i}{\partial y_j} \right\|_{i,j=1}^{i=3,j=9+2N}$$

where in the matrices A_*, B_* the derivatives are calculated on the unperturbed motion $\Omega_i^* = 0$, $\gamma_i^* = 0$, $\beta_i^* = 0$, $q^* = 0$, $q^{*\cdot} = 0$ ($i = 1, 2, 3,$), and we assume that

$$\det \left\| \frac{\partial U_i}{\partial y_j} \right\|_{i,j=1}^{i,j=3} \neq 0$$

If the quadratic form y^TAy is positive definite on a linear manifold $(B_*y)_i = b_{i1}y_1 + \ldots + b_{i,9+2N}y_{9+2N} = 0$ ($i = 1, 2, 3$), the values of the variables (2.8) will correspond to a local minimum of the integral c with conditions (1.6).

Introducing the determinant

$$\Delta = -\begin{vmatrix} \Theta & B \\ B_*^T & A_* \end{vmatrix} \tag{3.1}$$

where Θ is the 3×3 zero matrix, the necessary and sufficient conditions for this quadratic form to be positive definite on the liner manifold can be formulated as the condition of strict positivity [5]

$$\Delta_7 > 0, \Delta_8 > 0, \ldots, \Delta_{9+2N} \equiv \Delta > 0 \tag{3.2}$$

where Δ_n is the principal diagonal minor of order n ($n = 7, \ldots, 9+2N$).

If we expand the determinant Δ using Laplace's theorem [8] with respect to the first three rows, and then with respect to the first three columns, conditions (3.2) can be replaced by the equivalent Silvester determinant conditions of the positive definiteness of a matrix

$$H = \begin{Vmatrix} A & B \\ B^T & \omega^{-2}C \end{Vmatrix}$$

$$A = \operatorname{diag}(J^{kk}(q^0) - J^{mm}(q^0),\ J^{ll}(q^0) - J^{mm}(q^0),\ J^{kk}(q^0) - J^{ll}(q^0))$$

$$B^T \equiv (x_*, y_*, z_*), \quad x_* \equiv 2(J_1^{km}(q^0), \ldots, J_N^{km}(q^0))^T$$

$$y_* \equiv \sqrt{3}(J_1^{lm}(q^0), \ldots, J_N^{lm}(q^0))^T, \quad z_* \equiv (J_1^{kl}(q^0), \ldots, J^{kl}(q^0))^T$$

The positive definite matrix C was introduced in Sec. 1.

The $N \times N$ matrix $S \equiv \omega^2 C^{-1} = \|s_{ij}\|$ is positive definite and symmetrical [9].

Suppose $\alpha \in (0, 1)$ such that $J^{ll}(q^0) - J^{mm}(q^0) = (1-\alpha)(J^{kk}(q^0) - J^{mm}(q^0))$, $J^{kk}(q^0) - J^{ll}(q^0) \equiv \alpha(J^{kk}(q^0) - J^{mm}(q^0))$.

We will denote by a_1, a_2 and a_3 the roots of the cubic equation

$$t^3 - t^2p + tg - r = 0 \tag{3.3}$$

$$p = (y, y) + (x, x) + (z, z)$$

$$g = (x, x)(y, y) - (x, y)^2 + (y, y)(z, z) - (y, z)^2 + (x, x)(z, z) - (x, z)^2$$

$$r = \begin{vmatrix} (x, x) & (x, y) & (x, z) \\ (x, y) & (y, y) & (y, z) \\ (x, z) & (y, z) & (z, z) \end{vmatrix}$$

Here and below

$$(x, x) \equiv x_*^T S x_*, \quad (y, y) \equiv (1-\alpha)^{-1} y_*^T S y_*, \quad (z, z) \equiv \alpha^{-1} z_*^T S z_*$$

$$(x, y) \equiv (1-\alpha)^{-1/2} x_*^T S y_*, \quad (x, z) \equiv \alpha^{-1/2} x_*^T S z_*$$

$$(y, z) \equiv \alpha^{-1/2}(1-\alpha)^{-1/2} y_*^T S z_*$$

The larger root of the quadratic equation $(t-(x,x))(t-(y,y))-(x,y)^2=0$ will be denoted by a_*, $a_* = 0.5((x,x)+(y,y))+(0.25((x,x)+(y,y))^2+(x,y)^2)^{1/2}$.

We will assume [8] that $a_1 \leqslant a_2 \leqslant a_3$ if Eq. (3.3) has three different (or a multiple of three) real roots, when there are two multiple roots we will denote the simple one by a_3, and when Eq. (3.3) has one real root, we will denote it by a_3.

Using the well-known formula [9] for determinants $\det H = \det(\omega^{-2}C) \times \det(A - BSB^T)$ we obtain the following assertion.

Assertion 3.1. To satisfy conditions (3.2) it is necessary and sufficient that

$$J^{kk}(q^0) > J^{ll}(q^0) > J^{mm}(q^0) > 0 \tag{3.4}$$

$$J^{kk}(q^0) - J^{mm}(q^0) > \max(a_*, a_3) \tag{3.5}$$

if Eq. (3.3) has one real root (it can be of multiplicity three) a_3, otherwise

$$J^{kk}(q^0) - J^{mm}(q^0) > a_*$$

$$a_1 < J^{kk}(q^0) - J^{mm}(q^0) < a_2 \quad \text{or} \quad J^{kk}(q^0) - J^{mm}(q^0) > a_3 \tag{3.6}$$

$$\det(\| d_{ij}(q^0) \|_{i,j=1}^{i,j=n}) > 0 \quad (n = 1, \ldots, 3+N) \tag{3.7}$$

$$\det(\| c_{ij} \|_{i,j=1}^{i,j=p}) > 0 \quad (p = 1, \ldots, N)$$

Conditions (3.4) can be regarded as the sufficient conditions for the stability of an elastic spacecraft frozen in a position of equilibrium (2.8) [5, 7]. Conditions (3.7) are the Silvester determinant criterion of the positive definiteness of the matrices $D(q^0)$ and C, respectively. Without investigating the roots of Eq. (3.3) we can conclude that when the number of tones taken into account is increased (when N is increased), conditions (3.5) and (3.6) become more rigid (the value of a_* increases).

In addition to conditions (3.4)–(3.7) we will give the sufficient conditions for (3.2) to be satisfied, which is a consequence of a well-known theorem [9].

Assertion 3.2. For conditions (3.2) to be satisfied it is sufficient that H should be a matrix with a strict diagonal predominance, i.e.

$$J^{kk}(q^0) - J^{mm}(q^0) > 2\sum_{n=1}^{N} |J_n^{km}(q^0)|, \quad J^{kk}(q^0) - J^{ll}(q^0) > \sum_{n=1}^{N} |J_n^{kl}(q^0)| \tag{3.8}$$

$$J^{ll}(q^0) - J^{mm}(q^0) > \sqrt{3}\sum_{n=1}^{N} |J_n^{lm}(q^0)|$$

$$\frac{c_{ii}}{\omega^2} > \sum_{j=1}^{i-1} \frac{|c_{ij}|}{\omega^2} + 2|J_i^{km}(q^0)| + \sqrt{3}\,|J_i^{lm}(q^0)| + |J_i^{kl}(q^0)| \quad (i = 1, \ldots, N) \tag{3.9}$$

$$\det(\| d_{ij}(q^0) \|_{i,j=1}^{i,j=n}) > 0 \quad (n = 1, \ldots, N+3) \tag{3.10}$$

4. EXAMPLE

Suppose the spacecraft is modelled as a solid body to which is attached a uniform elastic rod of unit length and constant circular cross-section F at one end in a rectilinear and undeformed state, where the rest mass of the rod is $\tau = F\rho_1$ and ρ_1 is the density of the rod.

The unit vector F_1 of the axis of the undeformed rod is specified by the direction cosines with respect to the $Ox_1x_2x_3$ axes, $\mathbf{f}_1 = a\mathbf{i}^1 + b\mathbf{i}^2 + c\mathbf{i}^3$, $a^2+b^2+c^2 = 1$. Hence, the axis of the rod does not coincide with any of the principal central axes of the spacecraft in the undeformed state.

The radius vector of an arbitrary point of the axis of the undeformed rod is given by the expression $\mathbf{r}(s) = (r^0+s)\mathbf{f}_1 = r_1\mathbf{i}^1 + r_2\mathbf{i}^2 + r_3\mathbf{i}^3 = (r^0+s)a\mathbf{i}^1 + (r^0+s)b\mathbf{i}^2 + (r^0+s)c\mathbf{i}^3$, where r^0 is the distance from the point O_1 to the fastening of the end of the rod. We will assume that during deformation the rod executes small longitudinal-bending vibrations so that its cross-sections remain planar and perpendicular to the undeformed axis (Kirchhoff's hypothesis) [4].

In a local system of coordinates the unit vectors $\mathbf{f}_j$ $(j = 1, 2, 3)$

$$\mathbf{u}(t, s) \equiv w\mathbf{f}_1 + u\mathbf{f}_2 + v\mathbf{f}_3 = \sum_{n=1}^{h} (q_n^1 w_n \mathbf{f}_1 + q_n^2 u_n \mathbf{f}_2 + q_n^3 v_n \mathbf{f}_3) \tag{4.1}$$

where the natural forms of its free elastic vibrations [3]

$$w_n = \sin(\delta_n^0 s)$$
$$u_n \equiv v_n = (\sin\delta_n^* \operatorname{ch}\delta_n^* - \cos\delta_n^* \operatorname{sh}\delta_n^*)^{-1}((\operatorname{sh}\delta_n^* + \sin\delta_n^*)(\operatorname{ch}\delta_n^* s - \cos\delta_n^* s) -$$
$$-(\operatorname{ch}\delta_n^* + \cos\delta_n^*)(\operatorname{sh}\delta_n^* s - \sin\delta_n^* s))$$

(δ_n^0 is the nth root of the equation $\cos\delta = 0$) while δ_n^* is the nth root of the equation $\operatorname{ch}\delta\cos\delta + 1 = 0$).

We will assume that for finite rotation around an axis specified by the unit vector $\mathbf{i}^1 \times \mathbf{f}_1/|\mathbf{i}^1 \times \mathbf{f}_1|$ $(\mathbf{i}^1 \times \mathbf{f}_1 \neq 0)$ by an angle $\chi < 180°$, $\cos\chi = a$, and the unit vectors $\mathbf{i}^k$ change into $\mathbf{f}_k$ $(k = 1, 2, 3)$. We give below a table of the direction cosines of the unit vectors f_k in the $O_1x_1x_2x_3$ axes [1]

	$\mathbf{i}^1$	$\mathbf{i}^2$	$\mathbf{i}^3$
$\mathbf{f}_1$	$a \equiv g_{11}$	$b \equiv g_{21}$	$c \equiv g_{31}$
$\mathbf{f}_2$	$-b \equiv g_{12}$	$1 - \dfrac{b^2}{1+a} \equiv g_{22}$	$-\dfrac{cb}{1+a} \equiv g_{32}$
$\mathbf{f}_3$	$-c \equiv g_{13}$	$-\dfrac{cb}{1+a} \equiv g_{23}$	$1 - \dfrac{c^2}{1+a} \equiv g_{33}$

In the $O_1x_1x_2x_3$ axes, expression (1.1) has the form

$$\mathbf{u}(t, s) = \sum_{n=1}^{N} q_n(t)\psi_n(s) \tag{4.2}$$

where for each $j = 1, \ldots, h$, $i = 1, 2, 3$ we assume $p = i + 3(j-1)$

$$q_p \equiv q_j^i, \quad \psi_p \equiv (g_{1i}w_j)\mathbf{i}^1 + (g_{2i}u_j)\mathbf{i}^2 + (g_{3i}v_j)\mathbf{i}^3 \tag{4.3}$$

Henceforth we will confine ourselves to considering three terms in (4.2), assuming $h = 1$ in (4.1). In this case

$$\Pi = 0{,}5(c_{11}q_1^2 + c_{22}q_2^2 + c_{33}q_3^2) \tag{4.4}$$

Using formulas (4.3), (1.3) and (2.6) from (2.8) with $m = 3$, $k = 2$ and $l = 1$ we obtain

$$\begin{aligned} q_1^0 &= -\omega^2 c_{11}^{-1}(g_{21}^2 - 3g_{31}^2)I_w \\ q_2^0 &= -\omega^2 c_{22}^{-1}(4g_{22} - 3g_{11})I_u g_{21} \\ q_3^0 &= -\omega^2 c_{33}^{-1}(g_{11} - 4g_{33})I_v g_{31} \\ I_\kappa &= \int_0^1 \tau(r^0 + s)\kappa\, ds, \quad \kappa = u_1, v_1, w_1 \end{aligned} \tag{4.5}$$

Formulas (2.7) in this example have the form

$$\mathbf{e}^1(q^0) = \mathbf{i}^1 - \frac{P_{12}}{\mu^1 - \mu^2}\mathbf{i}^2 - \frac{P_{13}}{\mu^1 - \mu^3}\mathbf{i}^3$$

$$\mathbf{e}^2(q^0) = \frac{P_{12}}{\mu^1 - \mu^2}\mathbf{i}^1 + \mathbf{i}^2 - \frac{P_{23}}{\mu^2 - \mu^3}\mathbf{i}^3$$

$$e^3(q^0) = \frac{P_{13}}{\mu^1 - \mu^3} i^1 + \frac{P_{23}}{\mu^2 - \mu^3} i^2 + i^3$$

$$P_{12} = 2q_1^0 g_{11} g_{21} I_w + q_2^0 (g_{21} g_{12} + g_{11} g_{22}) I_u + q_3^0 (g_{21} g_{13} + g_{13} g_{23}) I_v$$
$$P_{13} = 2q_1^0 g_{11} g_{31} I_w + q_2^0 (g_{31} g_{12} + g_{11} g_{32}) I_u + q_3^0 (g_{11} g_{23} + g_{13} g_{21}) I_v$$
$$P_{23} = 2q_1^0 g_{21} g_{31} I_w + q_2^0 (g_{31} g_{22} + g_{32} g_{21}) I_u + q_3^0 (g_{23} g_{31} + g_{21} g_{33}) I_v$$

Assertion 2.2 can be made more specific for this example as follows.

Assertion 4.1. With Assumptions 1–5, the elastic rod in a position of relative equilibrium will be undeformed if and only if it is situated along one of the principal central axes of the spacecraft in the natural state and simultaneously along the tangent to the unit vector. We will get the following assertion from Eq. (4.5).

Assertion 4.2. With Assumptions 1–5, if the elastic rod is situated along one of the principal central axes of the spacecraft in the undeformed state and is perpendicular to the plane of the orbit, then in a position of relative equilibrium the rod will be compressed, and if the rod is situated along the radius of the orbit to an attracting centre (from it), then in a position of relative equilibrium it will be compressed (elongated), and in both cases there will be no bending strains.

Note. Assertions 4.1 and 4.2 remain true if one takes into account any number of tones [modes] in (4.2). The "sufficient part" of Assertions 4.1 and 4.2 has been derived in a number of papers (see the review in [2–4]), and can be obtained without Assumptions 1 and 2.

For the example considered, in view of the complex nature of conditions (3.4)–(3.7) we will give the sufficient conditions for stability of the relative positions of equilibrium (3.8)–(3.10).

Suppose the axis of the rod is situated along one of the principal central axes of the spacecraft in the undeformed state and perpendicular to the plane of the orbit. We will assume that $m = 3$, $k = 2$, $l = 1$, $g_{11} = 0$, $g_{21} = 1$ and $g_{31} = 0$; then from (4.5) we have $q_2^0 = q_3^0 = 0$, $q_1^0 \neq 0$, $P(q^0) = E$. If we denote by A_1 the moment of inertia of the spacecraft in the undeformed state about the transversal at the point O, by A_2 the moment of inertia about an axis perpendicular to the plane of the orbit and passing through O, and by A_3 the moment of inertia about an axis directed along the radius-vector of the orbit, passing through O, then conditions (3.8)–(3.9) take the following form:

$$c_{11} > 0, \quad c_{22} > \omega^2 \sqrt{3} |I_u|, \quad c_{33} > \omega^2 |I_v|$$
$$A_1 - A_3 > 0, \quad A_2 - A_1 + \omega^2 c_{11}^{-1} I_w^2 > |I_u|, \quad A_2 - A_3 + \omega^2 c_{11}^{-1} I_w^2 > |I_v|$$

I wish to thank L. Yu. Anapolskii for suggesting the problem and for discussing the results.

REFERENCES

1. LUR'YE A. I., *Analytical Mechanics*. Fizmatgiz, Moscow, 1961.
2. RUBANOVSKII V. N., The stability of the steady-state motions of complex mechanical systems. *Itogi Nauki i Tekhniki*. Obshchaya Mekhanika, VINITI, Moscow, Vol. 5, pp. 62–134, 1982.
3. DOKUCHAYEV L. V., *Non-linear Dynamics of Spacecraft with Deformable Elements*. Mashinostroyeniye, Moscow, 1987.
4. NABIULLIN M. K., *Steady-state Motions and Stability of Elastic Satellites*. Nauka, Novosibirsk, 1990.
5. RUBANOVSKII V. N. and SAMSONOV V. A., *The Stability of Steady-state Motions in Examples and Problems*. Nauka, Moscow, 1988.
6. GEL'FAND I. M., *Lectures on Linear Algebra*. Gostekhizdat, Moscow and Leningrad, 1951.
7. BELETSKII V. V., *The Motion of an Artificial Satellite about its Centre of Mass*. Nauka, Moscow, 1965.
8. KUROSH A. G., *Course of Higher Algebra*. Gostekhizdat, Moscow and Leningrad, 1946.
9. HORN R. and JOHNSON C., *Matrix Analysis*. Mir, Moscow, 1989.

Translated by R.C.G.